JOURNAL OF DIFFERENTIAL EQUATIONS **49**, 208–217 (1983)

A Neutral FDE with Stable D-Operator Is Retarded

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Received November 4, 1981; revised March 17, 1982

A neutral functional differential equation with a linear, autonomous, and stable D-operator can be rewritten as a retarded functional differential equation with infinite delay. This fact can be used, e.g., in the study of the stability of the neutral equation and to simplify the proofs of some standard asymptotic results.

1. INTRODUCTION

Hale and Mayer [5] published a monograph in 1967 on the theory of a class of neutral functional differential equations, and since then this theory has developed rapidly. A substantial number of reults for the retarded equation

$$x'(t) = f(t, x_t) \qquad (t \geqslant 0) \tag{1.1}$$

with initial condition

$$x(t) = \varphi(t) \qquad (-r \leqslant t \leqslant 0) \tag{1.2}$$

have been generalized to the neutral equation

$$\frac{d}{dt} D(t, x_t) = f(t, x_t) \qquad (t \geqslant 0) \tag{1.3}$$

with the same initial condition (see, e.g., [3]). Here x_t is the translated function $x_t(s) = x(s+t)$ $(-r \leqslant s \leqslant 0)$, and the operator D is supposed to be "atomic at zero" [3, p. 273]. In the proof of local existence, uniqueness, and continuation of solutions, one can let the D-operator in (1.3) depend on t, and it can be nonlinear, but a large majority of the results with which I am familiar on the stability of (1.2) and (1.3) require D to be linear and autonomous (i.e., independent of t), or a small perturbation of a linear

0022-0396/83 $3.00

autonomous operator. When D is linear and autonomous, I prefer to write (1.3) in the form

$$\frac{d}{dt}(\mu * x)(t) = f(t, x_t) \qquad (t \geqslant 0), \tag{1.4}$$

where μ is a matrix-valued measure supported on $[-r, 0]$ with an invertible point mass at zero (cf. [3, 8]). Actually, with a few exceptions (see, e.g., [6, 8, 9]) one usually assumes even more, namely that the D-operator is stable in the sense of [3, p. 287]. In the sequel, I restrict my attention to equations of type (1.4), and in the asymptotic analysis I assume that the D-operator (or equivalently, the measure μ in (1.4)) is stable.

In his book [3, p. 319], Hale mentions another branch of the theory of functional differential equations which is of even more recent date namely, the retarded equations with infinite delay.[1] Formally one gets such an equation by taking $r = -\infty$ in the definition of x_t and in (1.2). Some basic results are listed in [3, p. 319]; other, more recent results are found, e.g., in [4, 8, and 9] (see also the reference list in [8]).

The purpose of this note is to show that neutral functional differential equations with a linear, autonomous, and stable D-operator can be rewritten as a retarded equation with infinite delay. This fact can be used, e.g., in the study of the stability of the neutral equation and to simplify the proofs of some standard asymptotic results.

2. The Basic Transformation

Formally, it is very easy to rewrite (1.4) as a retarded equation. Define

$$y = \mu * x, \tag{2.1}$$

and suppose that the convolution operator $\mu *$ can be inverted (in one way or another), so that we can write

$$x = \mu^{-1} * y. \tag{2.2}$$

Then y satisfies the equation

$$y'(t) = f(t, \mu^{-1} * y_t) \qquad (t \geqslant 0) \tag{2.3}$$

with initial condition

$$y(t) = \psi(t) \qquad (t \leqslant 0), \tag{2.4}$$

[1] Of course, people working in Volterra integral equations have studied equations with infinite delay for a long time, but usually from a slightly different point of view.

where

$$\psi(t) = \mu * \varphi(t) \qquad (t \leqslant 0). \tag{2.5}$$

Below I shall make this formal argument precise. Indeed, with an appropriate choice of state space, the mapping $\mu *$ is continuous and invertible, and (2.3) is a retarded equation (with infinite delay).

The key step above is the inversion of the operator $\mu *$. Except in trivial cases, this inversion is not possible in a state space with finite memory. On the other hand, in some state spaces with infinite memory the inversion is very simple.

As I mentioned in the introduction, the measure μ is supposed to have an invertible point mass at zero. Without loss of generality, assume that this point mass is the identity matrix (multiply (1.4) by the inverse of this point mass). Write μ in the form $\mu = \delta - \nu$, where δ is the identity point mass at zero, and extend ν to a measure on $[0, \infty)$ by defining $d\nu(t) = 0$ for $t > r$. Then, formally,

$$\mu^{-1} = (\delta - \nu)^{-1} = \delta + \nu + \nu * \nu + \nu * \nu * \nu + \cdots. \tag{2.6}$$

Actually, this series converges in the following sense: Choose α so small that

$$\int_0^\infty e^{\alpha t}\, d|\nu|(t) < 1, \tag{2.7}$$

where $|\nu|$ is the total variation measure of ν. Then (2.6) defines a (locally finite) measure μ^{-1} which satisfies

$$\int_0^\infty e^{\alpha t}\, d|\mu^{-1}|(t) < \infty. \tag{2.8}$$

Let M_α be the set of measures ν satisfying

$$\|\nu\|_\alpha = \int_0^\infty e^{\alpha t}\, d|\nu|(t) < \infty.$$

Then both μ itself (extended to $[0, \infty)$ by zero for $t > r$) and μ^{-1} belong to M_α, and $\mu * \mu^{-1} = \mu^{-1} * \mu = \delta$. For more details, see [8, especially the proof of Theorem 5.2].

Measures in M_α define convolution operators on an appropriately chosen state space. Let C_α be the space of continuous vector-valued functions φ on $(-\infty, 0]$ satisfying $\lim_{t\to-\infty} e^{\alpha t}\varphi(t) = 0$, with norm

$$\|\varphi\|_\alpha = \max_{t\leqslant 0} e^{\alpha t}\, |\varphi(t)|.$$

For every $\varphi \in C_\alpha$, define

$$\mu * \varphi(t) = \int_{[-r,0]} d\mu(s)\, \varphi(t-s) \qquad (t \leqslant 0),$$

$$\mu^{-1} * \varphi(t) = \int_{(-\infty,0]} d\mu^{-1}(s)\, \varphi(t-s) \qquad (t \leqslant 0).$$

Then the convolution operators $\mu *$ and $\mu^{-1} *$ map C_α continuously into itself, and $\mu^{-1} *$ is the inverse of $\mu *$. Again, for more details, see [8].

Usually, when one discusses Eqs. (1.2) and (1.4), one works in the state space of continuous functions C on $[-r, 0]$ with norm

$$\|\varphi\| = \max_{-r \leqslant s \leqslant 0} |\varphi(s)|,$$

but I prefer to choose may state space to be C_α. Extend φ in (1.2) in an arbitrary way to a function in C_α and replace (1.2) by

$$x(t) = \varphi(t) \qquad (t \leqslant 0). \tag{2.9}$$

Define $x_t(s) = x(t+s)$ $(t \geqslant 0,\ s \leqslant 0)$, and interpret $\mu *$ and f in (1.4) as operators on C_α (before applying f, restrict x_t to $[-r, 0]$). In the state space C_α Eqs. (2.1) and (2.2) define a bicontinuous transformation, and (1.4), (2.9) become equivalent to (2.3)–(2.5). The space C_α satisfies the basic local state space axioms of [4], and (2.3) and (2.4) define a retarded equation in C_α in the sense of [4].

In the local theory of (2.3), (2.4), the value of α plays no role, but if I want to study the asymptotic properties of (2.3), (2.4), then I have to take $\alpha \geqslant 0$. This is due to the fact that

$$\|x_t\|_\alpha = e^{-\alpha t} \sup_{s \leqslant t} |e^{\alpha s} x(s)|, \tag{2.10}$$

so if $\alpha < 0$ and $x \not\equiv 0$, then $\|x_t\|_\alpha \to \infty$ as $t \to \infty$ (the global state space axiom (γ_2) in [4] is equivalent to $\alpha > 0$ in this case). Clearly, the larger I can choose α in (2.8), the sharper asymptotic results I can expect to get. Condition (2.7) is suffcient for (2.8) to hold, but in many cases (2.7) is much too conservative; e.g., it might only give me negative values of α, and I want to have $\alpha \geqslant 0$. The problem of the largest possible value of α in (2.8) leads me to a concept mentioned in the introduction, namely, the "stability of the D-operator," or equivalently, the stability of the convolution operator $\mu *$.

As it is shown in [3], the "difference" equation

$$\mu * x(t) = 0 \quad (t \geqslant 0), \qquad x(t) = \varphi(t) \quad (-r \leqslant t \leqslant 0), \tag{2.11}$$

generates a semigroup $T(t)$ in the space C^0 of continuous functions φ on $[-r, 0]$ satisfying $\mu * \varphi(0) = 0$. One simply defines $T(t)\varphi$ to be $T(t)\varphi = x_t$ (restricted to $[-r, 0]$), where x is the solution of (2.11).

LEMMA 2.1. *The supremum of all numbers α satisfying* (2.8) *equals $-\beta$, where β is the order of the semigroup generated by difference equation* (2.11) *in C^0. In particular,* (2.8) *holds for some $\alpha > 0$ iff $\mu *$ is stable.*

In one sense the conclusion of Lemma 2.1 is very natural, because μ^{-1} is the "fundamental solution" of (2.11). One could of course also look at the semigroup generated by (2.11) in $C_\alpha^0 = \{\varphi \in C_\alpha \mid \mu * \varphi(0) = 0\}$ for an arbitrary α, $-\infty < \alpha < \infty$. By (2.10) and Lemma 2.1, the order of this semigroup is $\max\{-\alpha, \beta\}$, with β as in Lemma 2.1.

Proof of Lemma 2.1. In the following proof, let B_α denote the set of continuous functions ψ on $(-\infty, \infty)$ satisfying $\lim_{t \to -\infty} e^{\alpha t}\varphi(t) = 0$, and

$$\|\psi\|_\alpha = \sup_{-\infty < t < \infty} e^{\alpha t} |\psi(t)| < \infty.$$

By [8, Lemma 2.1], if $\mu \in M_\alpha$ and $\psi \in B_\alpha$, then $\mu * \psi \in B_\alpha$, and $\|\mu * \psi\|_\alpha \leqslant \|\mu\|_\alpha \|\psi\|_\alpha$ (where $\|\mu\|_\alpha$ is the norm of μ in M_α).

Suppose that $\mu^{-1} \in M_\alpha$ for some α. For each φ in (2.11), choose a function $\psi \in B_\alpha$ such that $\psi(t) = \varphi(t)$ $(-r \leqslant t \leqslant 0)$, and $\|\psi\|_\alpha \leqslant \gamma \|\varphi\|$, where $\gamma = \max\{1, e^{-\alpha r}\}$. Define $y(t) = 0$ $(t \leqslant 0)$, $y(t) = x(t) - \psi(t)$ $(t \geqslant 0)$, $f(t) = 0$ $(t \leqslant 0)$, $f(t) = -\mu * \psi(t)$ $(t \geqslant 0)$. Then $f \in B_\alpha$ with $\|f\|_\alpha \leqslant \|\mu\|_\alpha \|\psi\|_\alpha \leqslant \gamma \|\mu\|_\alpha \|\varphi\|$, and

$$\mu * y(t) = f(t) \qquad (-\infty < t < \infty).$$

Clearly then, $y = \mu^{-1} * f \in B_\alpha$, and $\|y\|_\alpha \leqslant \gamma \|\mu^{-1}\|_\alpha \|\mu\|_\alpha \|\varphi\|$. As $x(t) = y(t)$ for $t \geqslant 0$, this shows that the order of the semigroup generated by (2.11) is at most $-\alpha$, so $\beta \leqslant -\alpha$.

Conversely, I have to show that $\mu^{-1} \in M_\alpha$ for every $\alpha < -\beta$. Pick any $\alpha < -\beta$, and define $\psi(t) = e^{\alpha t}\varphi(t)$ $(-r \leqslant t \leqslant 0)$, $y(t) = e^{\alpha t}x(t)$ $(t \geqslant -r)$, $d\mu_\alpha(t) = e^{\alpha t}\, d\mu(t)$ $(t \geqslant 0)$. Then (2.11) is turned into

$$\mu_\alpha * y(t) = 0 \quad (t \geqslant 0), \qquad y(t) = \psi(t) \quad (-r \leqslant t \leqslant 0). \tag{2.12}$$

This is an equation of the same type as (2.11), and the semigroup generated by (2.12) has order $\alpha + \beta < 0$, so $\mu_\alpha *$ is a stable operator according to [3, Theorem 4.1, p. 287]. Observe that if I define μ_α^{-1} in the same way as μ^{-1}, then $d\mu_\alpha^{-1}(t) = e^{\alpha t}\, d\mu^{-1}(t)$. This means that $\mu^{-1} \in M_\alpha$ iff $\mu_\alpha^{-1} \in M$, where $M = M_\alpha$ with α replaced by 0. Thus, to complete the proof it suffices to show that $\mu_\alpha^{-1} \in M$. To simplify the notations, I drop the subindex α, return to (2.11), assume that $\mu\,*$ is stable, and show that $\mu^{-1} \in M$.

Let B be the space B_α with α replaced by 0, with norm $\|\ \| = \|\ \|_0$. Pick an arbitrary function $h \in B$ with $h(t) = 0$ for $t \leqslant 0$. Then by [3, Theorem 4.1, p. 287], the solution y of

$$\mu * y(t) = h(t) \qquad (t \geqslant 0), \tag{2.13}$$

which vanishes for $t \leqslant 0$ satisfies

$$\|y\| \leqslant K\,\|h\| \tag{2.14}$$

for some constant K, independent of h. Translate y and h to the left by the same amount to see that for every $h \in B$ vanishing for $t \leqslant -T$, where T is an arbitrary number, I have a unique solution $y \in B$ of (2.13), vanishing for $t \leqslant -T$, and satisfying (2.14). This set of functions h is dense in B, and that implies that for every $h \in B$, I have a solution $y \in B$ of (2.13) satisfying (2.14). The solution operator $h \mapsto y$ is linear, it is continuous, and it commutes with translations, so it is induced by a matrix measure $\nu \in M$! On the other hand, when y and h vanish for $t \leqslant -T$, I can solve (2.13) by using μ^{-1}, and this shows that $\nu = \mu^{-1}$. *I* other words, $\mu^{-1} \in M$, and the proof of Lemma 2.1 is complete.

Let me summarize the preceding results into

PROPOSITION 2.2. *Let $\alpha > -\beta$, where β is the order of the semigroup generated by the difference equation* (2.11) *in* C^0. *Then the convolution operator $\mu\,*$ maps C_α continuously into itself, it is invertible, and its inverse is the operator $\mu^{-1}\,*$. If y is a solution of* (2.3), (2.4), *then $x = \mu^{-1} * y$ is a solution of* (1.2), (1.4), *with $\varphi(t) = \mu^{-1} * \psi(t)$ $(-r \leqslant t \leqslant 0)$, and conversely, if x is a solution of* (1.2), (1.4), *and φ is extended to a continuous function in C_α, then $y = \mu * x$ is a solution of* (2.3), (2.4), *with $\psi = \mu * \varphi$.*

By Proposition 2.2, the retarded equation (2.3), (2.4) is equivalent to the neutral equation (1.4), (2.9) in C_α. However, (1.2), (1.4) with state space C_α is not quite the same equation as (1.4), (2.9) with state space C. For the local theory, i.e., the theory on a finite time interval, this difference does not cause any problems, but one has to be more careful with the asymptotic theory. The space C_α with $\alpha < 0$ are useless for asymptotic theory (cf. the paragraph around Eq. (2.10)). On the other hand, for $\alpha > 0$, the norm in C_α is "fading," and asymptotically there is not much difference between (1.2), (1.4) with the state space C and (1.4), (2.9) with state space C_α. For instance, if u and v are bounded continuous solutions, then $\|u_t - v_t\|_\alpha \to 0$ as $t \to \infty$ iff $\|u_t - v_t\| \to 0$ as $t \to \infty$, because both conditions are equivalent to

$u(t) - v(t) \to 0$ as $t \to \infty$. A trajectory is precompact in C iff it is precompact in C_α (cf. [8, Lemma 2.6]). A function belongs to the limit set in C of the trajectory in C of a solution iff it is the restriction to $[-r, 0]$ of a function in the limit set in C_α of the trajectory in C_α of the same solution.

There is another way of rewriting (1.4) into a retarded equation which is even simpler, but slightly less general. If x is continuously differentiable for $t \geqslant -r$, then I can extend the initial function φ to a continuously differentiable function on $(-\infty, 0]$, vanishing for $t \leqslant -r-1$, and apply μ^{-1} directly to (1.4). This gives me the retarded equation

$$x'(t) = \mu^{-1} * g(t, x_t) \qquad (t \geqslant 0), \tag{2.15}$$

where

$$\begin{aligned} g(t, x_t) &= \mu * \varphi'(t), \qquad && (t \leqslant 0), \\ &= f(t, x_t), && (t \geqslant 0). \end{aligned} \tag{2.16}$$

The initial condition is (2.9) rather than (1.2). Equation (2.15) is of the same type as (2.3).

Usually a solution x of (1.4) is not continuously differentiable, but fortunately, one can make sense out of (2.15), (2.16) even under a much weaker assumption. If φ is absolutely continuous on $[-r, 0]$, then I can extend φ to an absolutely continuous function on $(-\infty, 0]$, vanishing for $t \leqslant -r-1$, and the same argument applies. The only difference is that x is locally absolutely continuous instead of continuously differentiable, and that (2.15), (2.16) hold only almost everywhere. In this case one can, e.g., replace C_α by the space of functions φ satisfying $\varphi \in L^1_\alpha$, $\varphi' \in L^1_\alpha$, where L^1_α is the space of integrable functions on $(-\infty, 0)$ with respect to the weight function $e^{\alpha t}$. If φ' is square integrable, then I can even work in the Hilbert space of functions φ satisfying $\varphi \in L^2_\alpha, \varphi' \in L^2_\alpha$, where L^2_α is defined analogously to L^1_α. The convolution operators $\mu *$ and $\mu^{-1} *$ behave just as well in these spaces as in C_α. Again, for more details, see, e.g., [8].

3. Applications and Extensions of the Basic Transformation

The basic transformation (2.1), (2.2) has not been used explicitly very much in the study of neutral functional differential equations with linear, autonomous, and stable D-operators. Datko [2, p. 118] uses this transformation for a D-operator of the form

$$D(t, \varphi) = \varphi(0) - \sum_{j=1}^{m} B_j(t)\, \varphi(t - h_j)$$

(which need not be autonomous), assuming something [2, line (1.24)] which in the autonomous case essentially amounts to $\mu^{-1} \in M_\alpha$ for some $\alpha > 0$. In his book [3], Hale develops the basic asymptotic theory for equations of this type without explicit use of the transformation (2.1), (2.2). However, Hale instead applies [3, Theorems 3.1, p. 281, and 4.1(iii), p. 287], which in spirit are very close to Lemma 2.1 above (cf. my proof of Lemma 2.1). Note in particular that the function y defined by (2.1) plays a decisive role in the formulation of the Liapunov and Razumikin theory [3, Sect. 12.7]. In some recent papers on (1.4) in an L^p-space [1, 8], the importance of the function y in (2.1) has been recognized, but there the inverse transformation (2.2) is not used.

One can simplify some of Hale's proofs by applying Lemma 2.1 rather than [3, Theorem 3.1(iii)]. For instance, suppose that $\mu *$ is stable, so that $\mu^{-1} \in M_\alpha$ for some $\alpha > 0$. Let f be bounded and continuous, and let x be a bounded and continuous solution of

$$\frac{d}{dt}(\mu * x)(t) = f(t, x_t) \qquad (-\infty < t < \infty).$$

Define y by (2.1). Then

$$y'(t) = f(t, \mu^{-1} * y_t) \qquad (-\infty < t < \infty),$$

so trivially, y' is bounded and continuous. As $x' = \mu^{-1} * y'$, also x' is bounded and continuous. This proves the case $k = 0$ of [3, Theorem 6.2, p. 294]. (The general case can be proved in a similar way.)

One can sometimes use the basic transformation to study the asymptotic behavior of a neutral equation by reducing it to a retarded equation. For instance, consider the scalar equation

$$x'(t) - ax'(t-1) = bg(x(t)) + \int_{-\infty}^{t} c(t-s)\, g(x(s))\, ds \qquad (t \geqslant 0), \quad (3.1)$$

with initial condition (2.9). Assume that $-1 < a < 1$, $b \leqslant 0$, c is integrable on $(0, \infty)$, and the initial function φ is bounded and locally absolutely continuous with φ' bounded. Clearly $\mu^{-1} = \sum_{k=0}^{\infty} a^k \delta_k$, where δ_k is the unit point mass at k. This time I use the second transformation mentioned in Section 2, i.e., I convolve (3.1) with μ^{-1} to get

$$\begin{aligned} x'(t) = b \sum_{k=0}^{[t]} a^k g(x(t-k)) + \int_0^t d(t-s)\, g(x(s))\, ds \\ + h(t) + f(t) \quad (t \geqslant 0), \quad x(0) = \varphi(0), \end{aligned} \qquad (3.2)$$

where $[t]$ is the integral part of t and

$$d(t) = \sum_{k=0}^{[t]} a^k b(t-k),$$

$$h(t) = a^{[t]+1}\varphi'(t-[t]-1),$$

$$f(t) = \sum_{k=0}^{[t]} a^k \int_{t-k}^{\infty} c(s)\, g(\varphi(t-k-s))\, ds$$

for $t \geqslant 0$. Equation (3.2) is a Volterra equation which has been studied extensively. For instance, if h and f are integrable over $(0, \infty)$, $G(x) = \int_0^x g(x)\, dx \to \infty$ $(x \to \pm\infty)$, $\limsup_{x \to \pm\infty} |g(x)|/G(x) < \infty$, and the real part of the Fourier transform of the kernel in (3.2) is negative, then all solutions of (3.2) are bounded and tend pointwise to zero as $t \to \infty$ [7, Theorem 6.1–6.3]. Observe that h is automatically integrable and so is f if $\int_0^\infty t\,|c(t)|\, dt < \infty$. The Fourier transform of the kernel is $(b + \hat{c}(\omega))/\hat{\mu}(\omega)$ $(-\infty < \omega < \infty)$, where $\hat{c}(\omega) = \int_0^\infty e^{-i\omega t} c(t)\, dt$, and $\hat{\mu}(\omega) = 1 - ae^{-i\omega}$, so the real part of the transform of the kernel is negative iff

$$[1 - a\cos(\omega)]\left[b + \int_0^\infty \cos(\omega t)\, c(t)\, dt\right]$$

$$+ a\sin(\omega)\int_0^\infty \sin(\omega t)\, c(t)\, dt < 0, \qquad (-\infty < \omega < \infty).$$

The same basic idea, the inversion of the D-operator, can be used also in many other occasions where the D-operator is not autonomous. For instance, to solve the equation

$$\frac{d}{dt}[\mu * x(t) + g(t)] = f(t, x_t) \qquad (t \geqslant 0),$$

one can define $y = \mu * x + g$ and solve

$$y'(t) = f(t, \mu^{-1} * (y - g)_t) \qquad (t \geqslant 0)$$

(cf. [3, p. 302] where Hale uses a different transformation). Also the functional equation

$$\mu * x(t) = f(t, x_t) \qquad (t \geqslant 0),$$

$$x(t) = \varphi(t) \qquad (t \leqslant 0), \quad \mu * \varphi(0) = f(0, \varphi),$$

can be discussed with the same method (cf. [3, p. 306]). It is even possible to let the D-operator be (weakly) nonlinear. For instance, if D maps C_α into itself and is of the form $D = I + E$, where I is the identity operator (convolution with δ), and E is a contraction, then by the contraction mapping principle, D can be inverted (specialized to the linear autonomous case this is exactly the same argument which proves the sufficiency of (2.7)). If one has a D-operator which does not behave well in the state space C_α, then one could try to use some other state space of fading memory type.

Sometimes one can do even better by not inverting the whole D-operator, but only a part of it. For instance, in the linear autonomous case, say that μ is of the form

$$d\mu(t) = d\nu(t) + a(t)\, dt,$$

where a is of bounded variation. Then I can write (1.4) in the form

$$\frac{d}{dt}(\nu * x(t)) = -a' * x(t) + f(t, x_t) \qquad (t \geqslant 0),$$

where a' is the measure derivative of a. By choosing a carefully one can often achieve $\beta_\nu < \beta$, where β_ν is the order of the semigroup generated by (2.11) with μ replaced by ν, and I can use ν instead of μ in (2.1), (2.2).

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